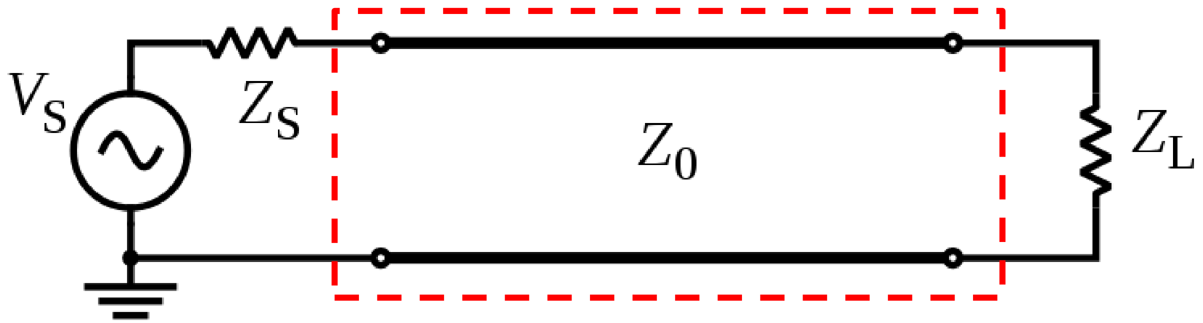
Waves on Transmission Lines

19TH OCTOBER 2020

Transmission lines

- A **transmission line** is used for the transmission of electrical power from generating substation to the various distribution units. It transmits the wave of voltage and current from one end to another.
- Any transmission line can be simply represented by a pair of parallel wires into one end of which power is fed by an a.c. generator.



Transmission line

https://en.wikipedia.org/wiki/Transmission_line#/media/File:Transmission_line_symbols.svg

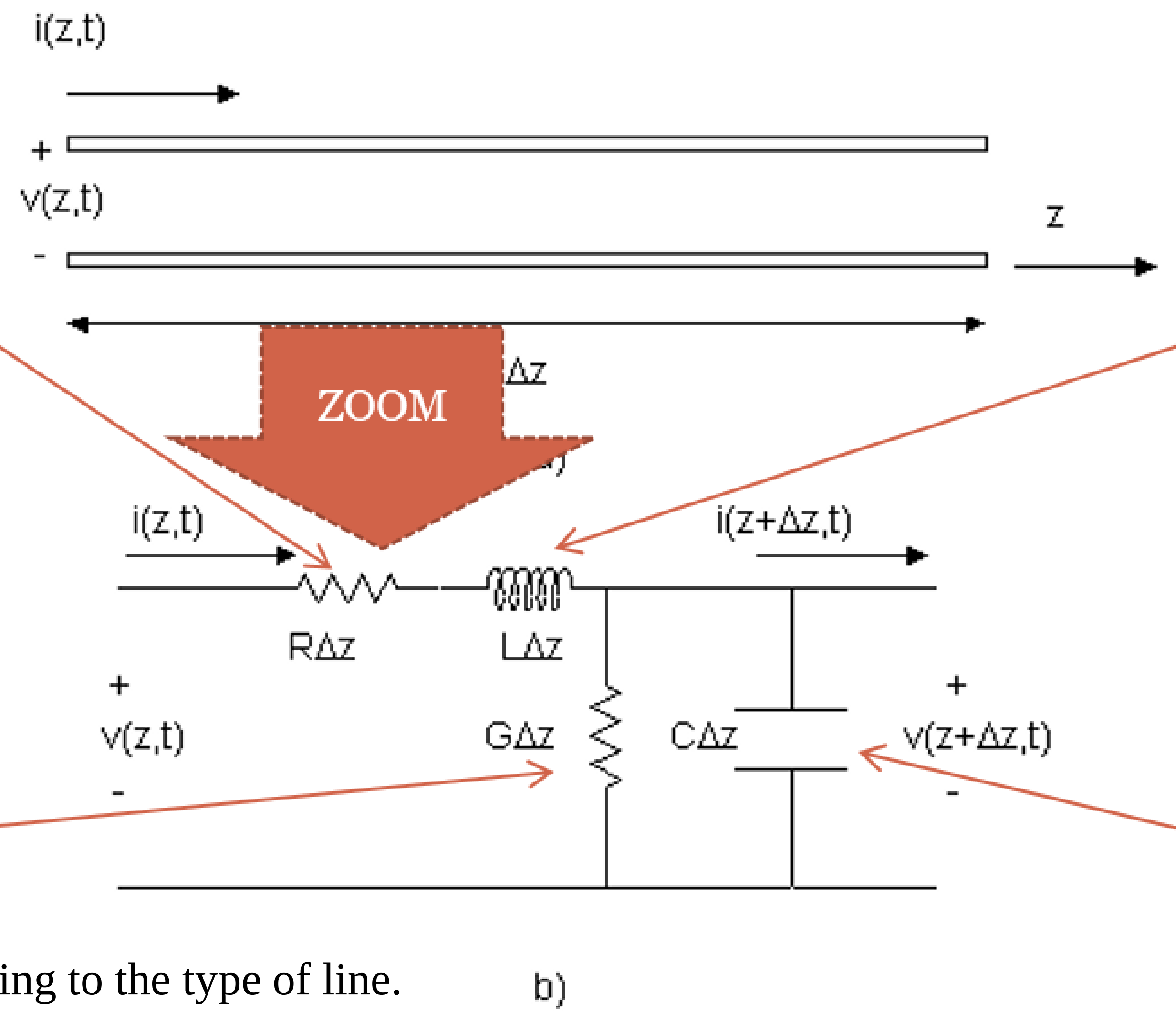
Wave equations for both voltage and current in a transmission line

- In the deriving the wave equation, a short element of the line is considered.
- Transmission line parameters are composed of **resistance**, **inductance**, **conductance** and **capacitance**.
- The flowing currents gives rise to the combined **inductance L_0 H per unit length**.
- Between the lines, which form a capacitor, there is an electrical **capacitance C_0 F per unit length**.
- Resistance in the line is represented by **R_0 Ω per unit length**.
- Conductance due to an imperfect insulating property of the insulator between the two conductors can be designated as **G_0 S per unit length**.

Distributed element in a transmission line

Distributed Resistance in the conductor: it models the power dissipation. **(Ω/Km)**

Distributed Inductance: produced by the magnetic field that surrounds the conductor. **(H/Km)**



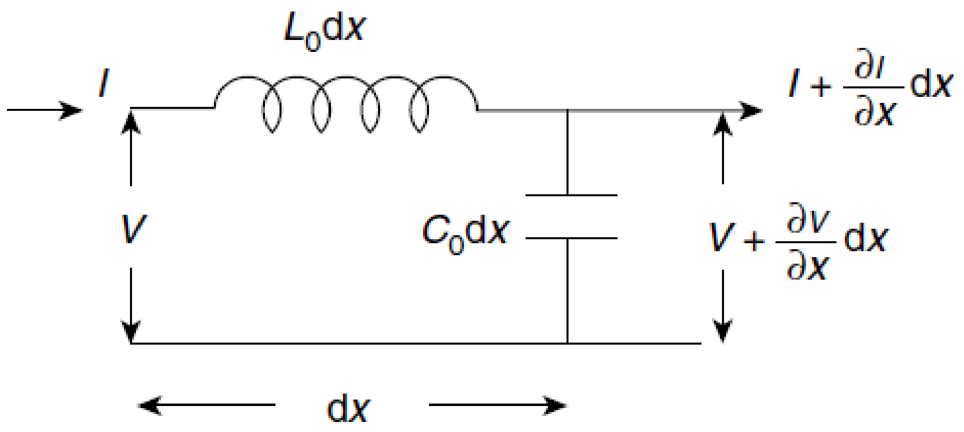
Distributed Condutance: it models the dielectric loses. **(S/Km)**

Distributed Capacitance: produced by the electric field existing in the dielectric between the conductors in the line. **(F/Km)**

These parameters vary according to the type of line.

Ideal or lossless transmission line

- In the absence of any resistance and conductance; i.e. $R_0 = G_0 = 0$, only L_0 and C_0 completely describe the line, which is known as **ideal or lossless**.



- Representation of element of an ideal transmission line length dx.
- The inductance of the element is L_0dx and capacitance of the element is C_0dx.

Derivation of the voltage and current waves

VOLTAGE WAVE EQUATION

- If the rate of change of voltage per unit length at constant time is ∂V/∂x.
- The voltage difference between the ends of the element dx is (∂V/∂x)dx, which is equals the voltage drop from the inductance -(L_0dx)∂I/∂t

$$\frac{\partial V}{\partial x}dx = -\left(L_0 dx\right)\frac{\partial I}{\partial t}$$

$$\therefore \frac{\partial V}{\partial x} = -L_0\frac{\partial I}{\partial t}$$

- Since $\partial^2/\partial x\partial t = \partial^2/\partial t\partial x$, a pure wave equation for the **voltage** with a velocity of propagation given by

$v^2 = 1/L_0C_0$ is

$$\frac{\partial^2 V}{\partial x^2} = L_0C_0\frac{\partial^2 V}{\partial t^2}$$

CURRENT WAVE EQUATION

- If the rate of change of current per unit length at constant time is ∂I/∂x, there is a loss of current along the length dx of -(∂I/∂x)dx.
- The loss is because some current has charged the capacitance C_0dx of the line to a voltage V.
- If the amount of charge is q = (C_0dx)V,

$$dI = \frac{dq}{dt} = \frac{\partial}{\partial t}\left(C_0 dx\right)V\text{; differential current through C}$$

- So that $-\frac{\partial I}{\partial x}dx = \frac{\partial}{\partial t}\left(C_0 dx\right)V \Rightarrow -\frac{\partial I}{\partial x} = \frac{\partial}{\partial t}C_0 V$

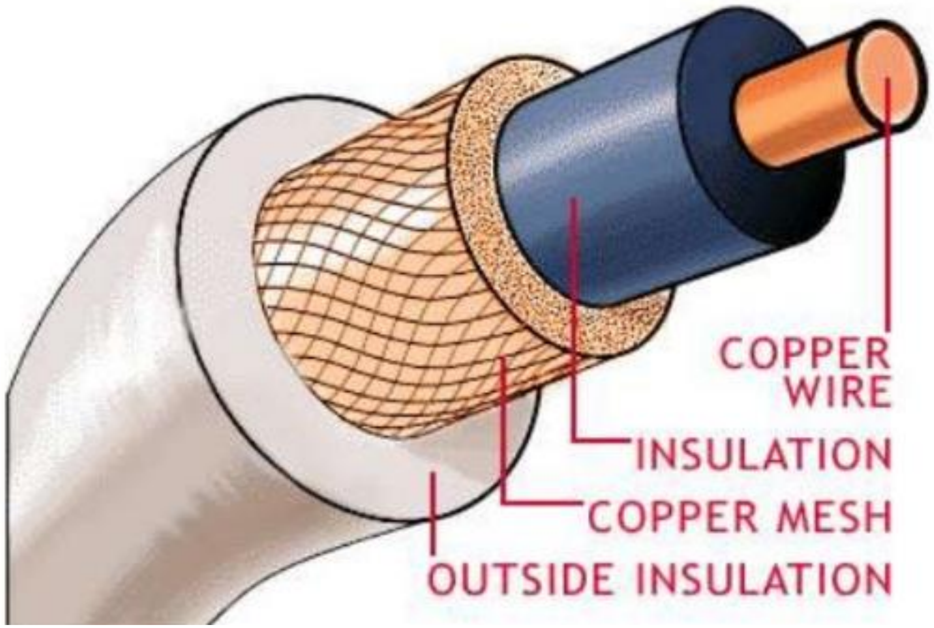
- Since $\partial^2/\partial x\partial t = \partial^2/\partial t\partial x$, a pure wave equation for the **current** with a velocity of propagation given by

$v^2 = 1/L_0C_0$ is

$$\frac{\partial^2 I}{\partial x^2} = L_0C_0\frac{\partial^2 I}{\partial t^2}$$

Coaxial cables

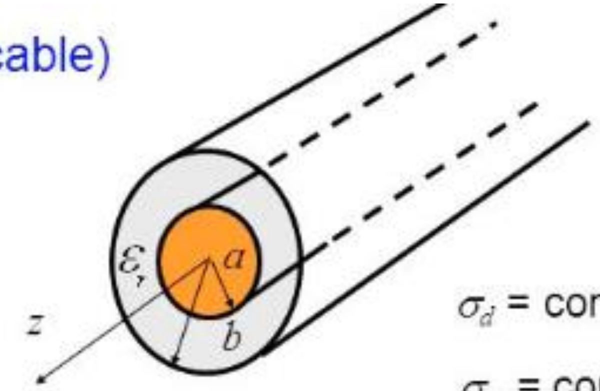
A diagram of a coaxial cable:



- Many transmission lines are made in the form of coaxial cables.
- The structure of the cables are composed of a cylinder of dielectric material having one conductor along its axis and the other surrounding its outer surface.

Inductance per unit length and conductance per unit length of coaxial cable

Example (coaxial cable)



σ_d = conductivity of dielectric [S/m].

σ_m = conductivity of metal [S/m].

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$$

$$\varepsilon_0 = (36\pi \times 10^9)^{-1}\,\mathrm{F\,m^{-1}}$$

$$C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\frac{b}{a}\right)} \quad [\mathrm{F/m}]$$

$$G = \frac{2\pi\sigma_d}{\ln\left(\frac{b}{a}\right)} \quad [\mathrm{S/m}]$$

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \quad [\mathrm{H/m}]$$

$$R = \frac{1}{\sigma_m\delta}\left(\frac{1}{2\pi a} + \frac{1}{2\pi b}\right) \quad [\Omega/\mathrm{m}]$$

https://slideplayer.com/slide/1631556/

Characteristic impedance Z_0

- The ratio of the voltage to the current in the waves travelling along the cable is

$$\frac{V}{I} = Z_0 = \sqrt{\frac{L_0}{C_0}}$$

- **Z_0 is defined as the characteristic impedance if the impedance is seen by the waves moving down *an infinitely long cable*.**

- The impedance of the coaxial cable can be written as

$$Z_0 = \sqrt{\frac{L_0}{C_0}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\,\mathbf{ln}\frac{b}{a} = \frac{1}{2\pi}\frac{1}{\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathbf{ln}\frac{b}{a}$$

where $\sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.6\ \Omega$ and $\mu_r \approx 1$

Characteristic impedance of a transmission line (1)

- Recall the wave equations of voltage and current waves

$$\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2} \qquad \frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2}$$

- The solutions for travelling wave propagating in the **positive direction** are givens as

Both travelling waves are in phase.

$$V_+ = V_{0+} \sin \frac{2\pi}{\lambda}(vt - x)$$

$$I_+ = I_{0+} \sin \frac{2\pi}{\lambda}(vt - x)$$

"+" refers to a wave moving in the positive x-direction.

- The voltage and current relates to each other via $(\partial V/\partial x) = -(L_0)\partial I/\partial t$: the voltage drops across an element length dx.
- This gives

$$V_+ = vL_0 I_+$$

https://www.computercablestore.com/1000ft-rg-59u-coaxial-cable-75-ohm-shielded-riser-cmr

Example data sheet of coaxial cable RG 59/U

Product Specifications

COMMSCOPE®

8261303/10 | 5571 WHRL RG59 40 1000

RG 59 Type 40% Braid Non-Plenum Coaxial Cable, white jacket, 1000 ft (305 m) reel

Construction Materials

Construction Type	Non-armored
Center Conductor Material	Copper-clad steel wire
Dielectric Material	Foam PE
Jacket Material	PVC
Shield (Braid) Coverage	40 %
Shield (Braid) Gauge	34 AWG
Shield (Braid) Material	Aluminum
Shield (Tape) Material	Aluminum/Poly

Electrical Specifications

Capacitance	54.8 pF/m | 16.7 pF/ft
Characteristic Impedance	75 ohm
Characteristic Impedance Tolerance	±3 ohm
Conductor dc Resistance	48.20 ohms/kft
Dielectric Strength, conductor to shield	2500 Vdc
Jacket Spark Test Voltage	4000 Vac
Nominal Velocity of Propagation (NVP)	82 %
Shield dc Resistance	14.90 ohms/kft
Structural Return Loss	20 dB @ 5–450 MHz
Structural Return Loss Test Method	100% Swept Tested

Dimensions

Cable Length	305 m | 1000 ft
Cable Weight	23.00 lb/kft
Diameter Over Center Conductor	0.8128 mm per 1 strand 0.0320 in per 1 strand
Diameter Over Dielectric	3.6576 mm | 0.1440 in
Diameter Over Dielectric Tolerance	±0.004 in
Diameter Over Jacket	6.147 mm | 0.242 in
Diameter Over Jacket Tolerance	±0.004 in
Diameter Over Shield (Braid)	4.521 mm | 0.178 in
Jacket Thickness	0.813 mm | 0.032 in
Jacket Thickness, minimum spot	0.635 mm | 0.025 in

subjected to nuclear radiation and does not have good high voltage characteristics. FEP is extrudable in a manner similar to PVC and polyethylene. This means that long wire and cable lengths are available. TFE is extrudable in a hydraulic ram type process. Lengths are limited due to amount of material in the ram, thickness of the insulation, and preform size. TFE must be extruded over a silver or nickel-coated wire. The nickel and silver-coated designs are rated 260°C and 200°C maximum, respectively.

	Halar®	Thermoplastic fluoropolymer material with excellent chemical resistance, electrical properties, thermal characteristics, and impact resistance. The temperature rating is -70°C to 150°C.
	Neoprene	The temperature range of this material can vary from -55°C to 90°C. The actual range would depend on the formulation used. Neoprene is both oil-resistant and sunlightresistant, making it ideal for many outdoor applications. The most stable colors are Black, Dark Brown, and Gray. The electrical properties are not as good as other insulation materials. Because of this, thicker insulation should be used. Typical designs where this material is used are lead wire insulation and cable jackets.
PE	**Polyethylene (Solid and Foam)**	A very good insulation in terms of electrical properties. Low dielectric constant, a stable dielectric constant over all frequencies, very high insulation resistance. In terms of flexibility, polyethylene can be rated stiff to very hard, depending on molecular weight and density: low density being the most flexible, and high-density, high-molecular weight formulation being very hard. Polyethylene also has an excellent moisture resistance rating. Correct Brown and Black formulations have excellent weather resistance. The dielectric constant is 2.3 for solid insulation and 1.64 for foam designs. Flame retardant formulations are available with dielectric constants ranging from about 1.7 for foam flame retardant to 2.58 solid flame retardant polyethylene.
	Polymer Alloy	Polymer Alloy is a plenum grade chloride-based jacketing material with low smoke and low flame spread properties. Cables jacketed with Polymer Alloy meet the UL Standard 910, Plenum Cable Flame Test.
PP	**Polypropylene (Solid and Foam)**	Similar in electrical properties to polyethylene, this material is primarily used as an insulation material. Typically, it is harder than polyethylene. This makes it suitable for thin wall insulations. UL maximum temperature rating may be 60°C maximum. The dielectric constant is 2.25 for solid and 1.55 for foam designs.
PU	**Polyurethane**	This material is used primarily as a cable jacket material. It has excellent oxidation, oil, and ozone resistance. Some formations also have good flame resistance. It is a sturdy compound with excellent properties, making it an ideal jacket material for retractile cords.
PVC	**Polyvinyl Chloride**	Sometimes referred to as vinyl. Extremely high or low temperature properties cannot be found in one formulation. Certain formulations may have -55°C to 105°C rating. Other common vinyls may have -20°C to 60°C. The many varieties of PVC also differ in flexibility and electrical properties. The price range can vary accordingly. Typical dielectric constant values can vary from 3.5 to 6.5. (FR-PVC - Fire Resistant Polyvinyl Chloride)

Key word :
foam PE dielectric constant

Calculation : Z_0, C_0 and wave speed

- **The characteristic impedance Z_0** of a coaxial cable

$$Z_0 = \sqrt{\frac{L_0}{C_0}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{b}{a} = \frac{1}{2\pi}\frac{1}{\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}}\ln\frac{b}{a} \text{ where } \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.6\ \Omega \text{ and } \mu_r \approx 1$$

- Given numerical data : ε_r = dielectric constant = 1.64, $\varepsilon_0 = (36\pi \times 10^9)^{-1}$ $\mathrm{Fm^{-1}}$, $\mu 0 = (4\pi \times 10^{-7})$ $\mathrm{Hm^{-1}}$

  $2b = 4.521$ mm and $2a = 0.8128$ mm

- Therefore, the characteristic impedance is found to be around 80 Ω.

- **The capacitance per unit length** : $C_0 = \frac{2\pi\varepsilon_r\varepsilon_0}{\ln\left(\frac{b}{a}\right)} = 53.1$ $\mathrm{pFm^{-1}}$

- Wave speed : $v^2 = 1/L_0C_0 = 1/\ \varepsilon_0\ \varepsilon_r\ \mu_0$. This gives $v \approx 2.34 \times 10^8$ m/s or 80% of light speed in vacuum.

Characteristic impedance of a transmission line (2)

- Subsequently, this gives $$\frac{V_+}{I_+} = vL_0 = \sqrt{\frac{L_0}{C_0}} = Z_0$$

- The value of Z_0 for the coaxial cable is found to be $$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\,\mathbf{ln}\,\frac{b}{a}$$

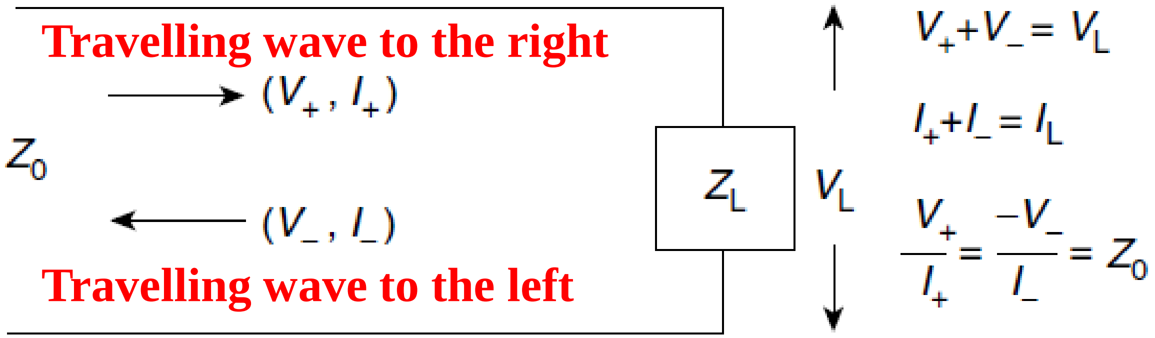
- **In common with the specific acoustic impedance, a negative sign is introduced to the ratio when the waves are travelling in the negative x-direction**.

- When waves are travelling in both directions along the transmission line, the total voltage and current at any point will be given by

$$V = V_+ + V_-$$
$$I = I_+ + I_-$$

Reflection from the end of a transmission line

- Suppose that transmission line of characteristic impedance Z_0 has a finite length and that the end opposite that of the generator is terminated by a load of impedance Z_L.



- Provided that the boundary condition at Z_L must be $V_+ + V_- = V_L$, where V_L is the voltage across the load and $I_+ + I_- = I_L$.
- In addition, $V_+/V_- = Z_0$, $V_-/I_- = -Z_0$ and $V_L/I_L = Z_L$.

Voltage amplitude coefficient

Current amplitude coefficient

Reflection coefficient

$$\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Transmission coefficient

$$\frac{V_L}{V_+} = \frac{2Z_L}{Z_L + Z_0}$$

Reflection coefficient

$$\frac{I_-}{I_+} = \frac{Z_0 - Z_L}{Z_L + Z_0}$$

Transmission coefficient

$$\frac{I_L}{I_+} = \frac{2Z_0}{Z_L + Z_0}$$

It is clear that if the line is terminated by a load $Z_L = Z_0$, its characteristic impedance, the line is matched, all the energy propagating down the line is absorbed and there is no reflected wave.

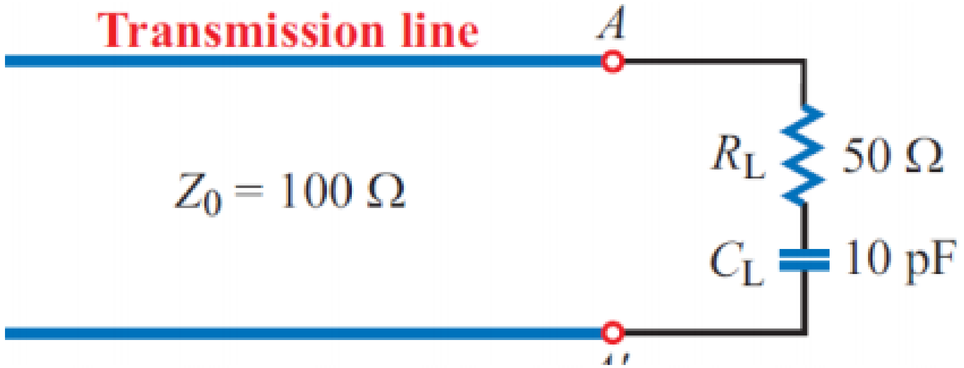
Derivation of the reflection coefficient

- Starting with $V_+ + V_- = V_L$ and $I_+ + I_- = I_L$, derive

$$\frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Example

A 100-Ω transmission line is connected to a load consisting of a 50-Ω resistor in series with a 10-pF capacitor. Find the reflection coefficient at the load for a 100-MHz signal.



Solution

$$\text{load impedance } Z_L = R - iX_c = R - i\frac{1}{2\pi fC}$$

$$= 50 - i\frac{1}{2\pi\left(10^8\right)\left(10^{-11}\right)}$$

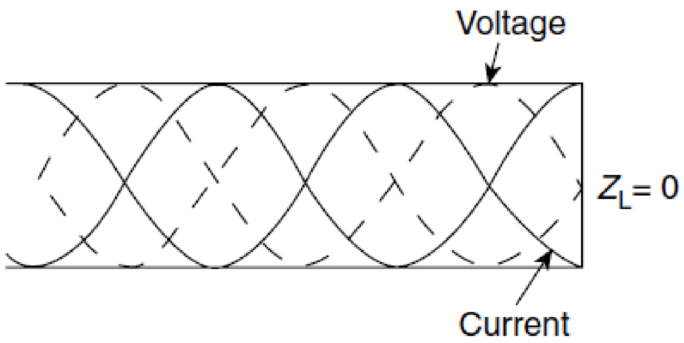
$$= 50 - i159$$

$$\text{reflection coefficient } \frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\left(50 - i159\right) - 100}{\left(50 - i159\right) - 100}$$

$$= 0.77e^{i\left(59^0\right)}$$

Short circuited transmission line ($Z_L = 0$)

- If the ends of the transmission line are short circuited, we have $V_L = V_+ + V_- = 0$
- This gives V+ = -V and there is total reflection with a phase change of π.



- This is the condition for the existence of standing waves.
- The figure shows that the voltage and current standing waves are **out of phase in space by 90^0**.
- In addition, both of them are also **out of phase by 90^0 in time**.

Derivation of the voltage and current standing waves at any point along the transmission line.

- At any position x on the line, the two voltage waves may be written as

$$V_+ = Z_0 I_+ = V_{0+} e^{i(\omega t - kx)}$$

$$V_- = \quad\quad\quad$$

+ direction

- direction

- With the total reflection and π phase shift, $V_{0+} = -V_{0-}$,
- The total voltage at x is $$V_x = (V_+ + V_-) = V_{0+}\left(e^{-ikx} - e^{ikx}\right)e^{i\omega t} = (-i)2V_0 e^{i\omega t}\,\mathbf{sin}\,kx$$
- The total current at x is $$I_x = (I_+ + I_-) = \frac{V_{0+}}{Z_0}\left(e^{-ikx} + e^{ikx}\right)e^{i\omega t} = \frac{2V_0}{Z_0}e^{i\omega t}\,\mathbf{cos}\,kx$$
- **Can you see the phase difference between voltage and current in space and time?**

Phase difference between current and voltage

- Recall total voltage and current at x : $V_x = (-i)2V_0 e^{i\omega t} \sin kx$

$$I_x = 2\frac{V_0}{Z} e^{i\omega t} \cos kx$$

- Phase difference in space

$$V_x space = \sin kx = \cos\left(kx - \frac{\pi}{2}\right)$$

$$I_x space = \cos kx$$

Current I leads voltage V by $\frac{\pi}{2}$.

- Phase difference in time

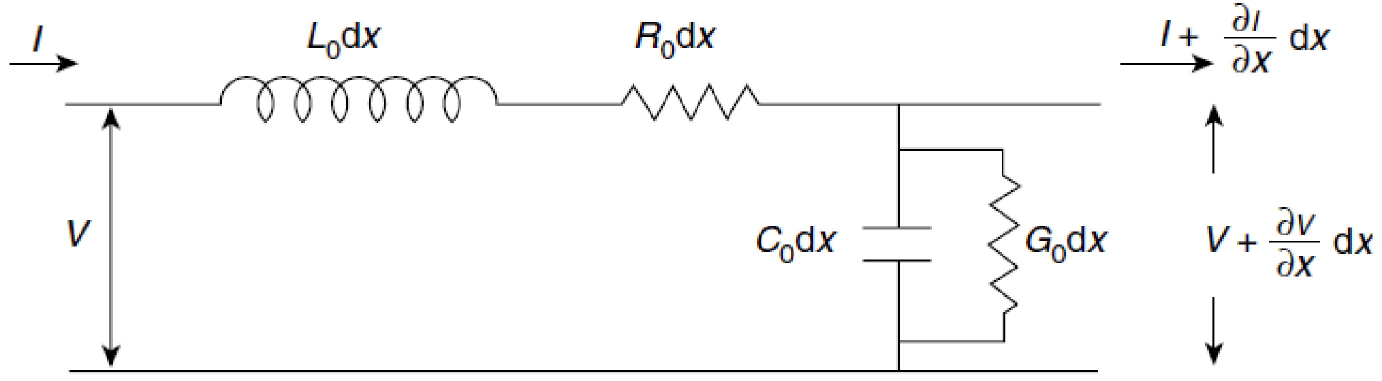
$$V_x time = -ie^{i\omega t} = e^{i\left(\omega t - \frac{\pi}{2}\right)}$$

$$I_x time = e^{i\omega t}$$

Current I still leads voltage V by $\frac{\pi}{2}$.

Effect of resistance in a transmission line

- In practice, some resistance always exists in the wires which will be responsible for energy losses.
- The transmission line is supposed to have a series resistance R_0 Ω per unit length and a short circuiting or shunting resistance between the wire.
- The inverse shunt resistance is represented as a shunt conductance G_0 (siemens per meter).
- **Current will now leak across the transmission line because the dielectric is not perfect**.



Real transmission line element includes a series resistance R_0 Ω per unit length and a shunt conductance G_0 S per unit length

Travelling waves in a transmission line with resistance (1)

- Recall the voltage and current changes across the line element length dx in case of **lossless line**,

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} \quad \text{and} \quad \frac{\partial I}{\partial x} = -\frac{\partial}{\partial t} C_0 V$$

- Now, adding the resistance R_0 and conductance G_0 to the equation,

Voltage drop across inductor

Voltage drop across resistor

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} - R_0 I = -\left(R_0 + i\omega_0 L_0\right) I,$$

Current drawn to capacitor

$$\frac{\partial I}{\partial x} = -\frac{\partial}{\partial t} C_0 V - G_0 V = -\left(G_0 + i\omega C_0\right) V; \text{where} \quad V = V_0 e^{i\omega t} \text{ and } I = I_0 e^{i\omega t}$$

Current drawn to shunt resistor

- Inserting ∂/∂x into one of the above equation gives

$$\frac{\partial^2 V}{\partial x^2} = -\left(R_0 + i\omega L_0\right)\frac{\partial I}{\partial x} = \left(R_0 + i\omega L_0\right)\left(G_0 + i\omega C_0\right) V = \gamma^2 V$$

where $\gamma^2 = \left(R_0 + i\omega L_0\right)\left(G_0 + i\omega C_0\right)$ which may be written as $\gamma = \alpha + ik$

Propagation constant (γ)

Attenuation or absorption coefficient (α)

Wave number (k)

Travelling waves in a transmission line with resistance (2)

- Similarly, the differential equation for the current may written as

$$\frac{\partial^2 I}{\partial x^2} = -\left(G_0 + i\omega C_0\right)\frac{\partial V}{\partial x} = \left(R_0 + i\omega L_0\right)\left(G_0 + i\omega C_0\right)I = \gamma^2 I$$

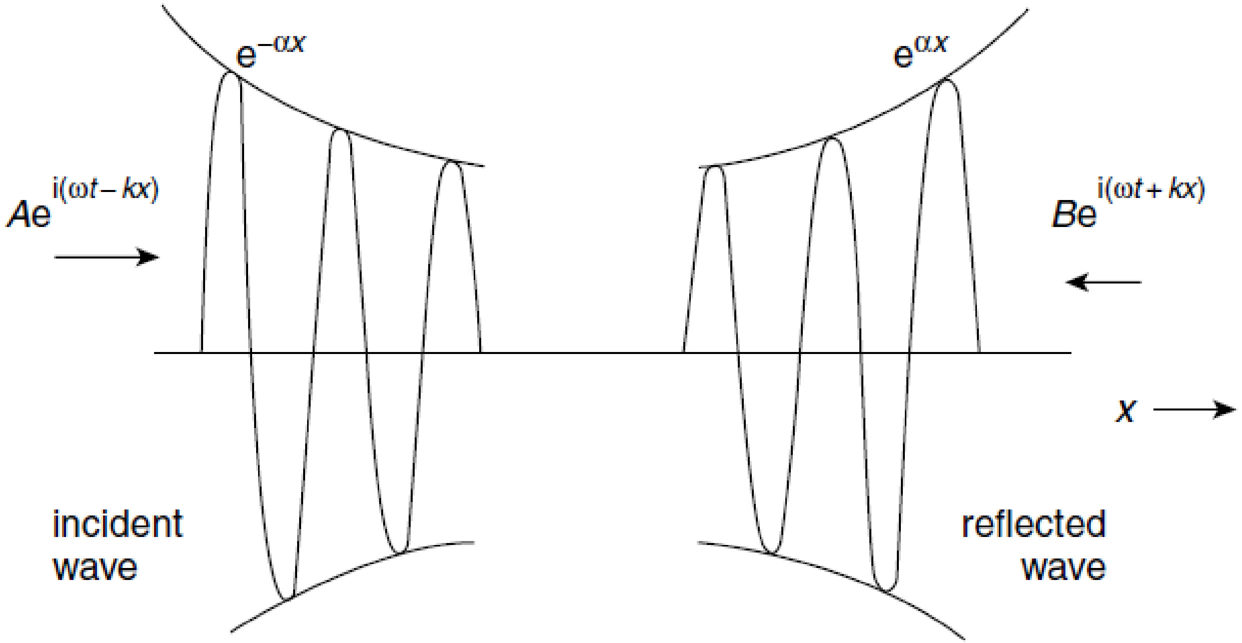
- It is clear that the solutions for x-dependence of $\frac{\partial^2 V}{\partial x^2} = \gamma^2 V$ are of the form

$V = Ae^{-\gamma x}$ or $V = Be^{+\gamma x}$;where A and B are constants.

- The complete solution with the time-dependence term exp(iωt) may be written as

$$V = \left(Ae^{-\gamma x} + Be^{+\gamma x}\right)e^{i\omega t} = \underline{Ae^{-\alpha x}}e^{i(\omega t - kx)} + \underline{Be^{\alpha x}}e^{i(\omega t + kx)}$$

Amplitude attenuation terms of travelling waves caused by the resistance in the transmission line



- Voltage and current waves in both directions along a transmission line with resistance.

  The effect of the dissipation term is shown by the exponentially decaying wave in each direction.
- Note that, the behavior of the current wave I is exactly similar.
- Since power is the product VI, the power loss with distance varies as $\left(e^{-\alpha x}\right)^2$

Characteristic impedance of a transmission line with resistance

- Let's consider one of the solution to the equation $\partial^2 I/\partial x^2 = \gamma^2 I$,

$I = A'e^{-\gamma x}$;the current wave in the positive x-direction.

- Recall $\frac{\partial I}{\partial x} = -(G_0 + i\omega C_0)V$, this leads to $-\gamma\left(A'e^{-\gamma x}\right) = -(G_0 + i\omega C_0)V$

- Or

$$\frac{\sqrt{(R_0 + i\omega L_0)(G_0 + i\omega C_0)}}{(G_0 + i\omega C_0)}\left(\underbrace{A'e^{-\gamma x}}_{I_+}\right) = V_+$$

$$\therefore \frac{V_+}{I_+} = \sqrt{\frac{(R_0 + i\omega L_0)}{(G_0 + i\omega C_0)}} = Z_0'$$

Similarly,

$$\therefore \frac{V_-}{I_-} = \sqrt{\frac{(R_0 + i\omega L_0)}{(G_0 + i\omega C_0)}} = -Z_0'$$

Wave equation with diffusion effects

- In case of having energy-loss mechanism in a wave propagation, the wave equation may be modified as

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{d}\frac{\partial \phi}{\partial t}$$

d = diffusivity with the dimension of length²/time

- Suppose the solution is given by $\phi = \phi_m e^{i(\omega t - \gamma x)}$.

- By finding $\frac{\partial^2 \phi}{\partial x^2}, \frac{\partial \phi}{\partial t}$ and $\frac{\partial^2 \phi}{\partial t^2}$ and substituting back into the modified wave equation, we have

$$i^2\gamma^2 = i^2\frac{\omega^2}{c^2} + i\frac{\omega}{d}$$

Or

$$\gamma^2 = \frac{\omega^2}{c^2} - i\frac{\omega}{d}$$

Wave equation with diffusion effects (contd.)

- If the propagation constant $\gamma = k - i\alpha$ where $\omega^2/c^2 = k^2$,

$$\gamma^2 = k^2 - 2ik\alpha - \alpha^2 \approx k^2 - 2ik\alpha \quad \text{if} \ \alpha \ll k$$

- The solution for ϕ then becomes

$$\phi = \phi_m e^{i(\omega t - \gamma x)} = \boxed{\phi_m e^{-\alpha x}} e^{i(\omega t - kx)}$$

- The solution shows that a sine or cosine oscillation of maximum amplitude ϕ_m decays exponentially with distance x.
- Diffusion mechanisms will cause attenuation or energy loss from the wave; the energy in a wave is proportional to the square of its amplitude and therefore decays as $e^{-2\alpha x}$.

Homework #8

Problem 7.5

Show that the impedance of a real transmission line seen from a position x on the line is given by

$$Z_x = Z_0 \frac{A\,e^{-\gamma x} - B\,e^{+\gamma x}}{A\,e^{-\gamma x} + B\,e^{+\gamma x}}$$

where γ is the propagation constant and A and B are the current amplitudes at $x = 0$ of the waves travelling in the positive and negative x-directions respectively. If the line has a length l and is terminated by a load Z_L, show that

$$Z_L = Z_0 \frac{A\,e^{-\gamma l} - B\,e^{\gamma l}}{A\,e^{-\gamma l} + B\,e^{\gamma l}}$$

Problem 7.6

Show that the input impedance of the line of Problem 7.5; that is, the impedance of the line at $x = 0$, is given by

$$Z_i = Z_0 \left(\frac{Z_0 \sinh \gamma l + Z_L \cosh \gamma l}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l} \right)$$

$$\left(\textit{Note}: 2\cosh \gamma l = e^{\gamma l} + e^{-\gamma l} \quad 2\sinh \gamma l = e^{\gamma l} - e^{-\gamma l} \right)$$

Problem 7.7

If the transmission line of Problem 7.6 is short-circuited, show that its input impedance is given by

$$Z_{sc} = Z_0 \tanh \gamma l$$

and when it is open-circuited the input impedance is

$$Z_{0c} = Z_0 \coth \gamma l$$

By taking the product of these quantities, suggest a method for measuring the characteristic impedance of the line.